APPENDIX B: LESSON PLANS

Lesson Plan I

Topic - Linear Transformations in Two Dimension
Sub Topic - Introduction to Linear Transformations

Specific Objectives:
Students would be able to represent regions and vectors in 2-dimensions in terms of matrices.
Students would be able to translate between matrix (algebraic) and geometrical representations of mathematical objects.

Pre Requisite Knowledge:
Students must be familiar with Coordinate Geometry & Matrix Multiplication and have knowledge about the structure of matrices.

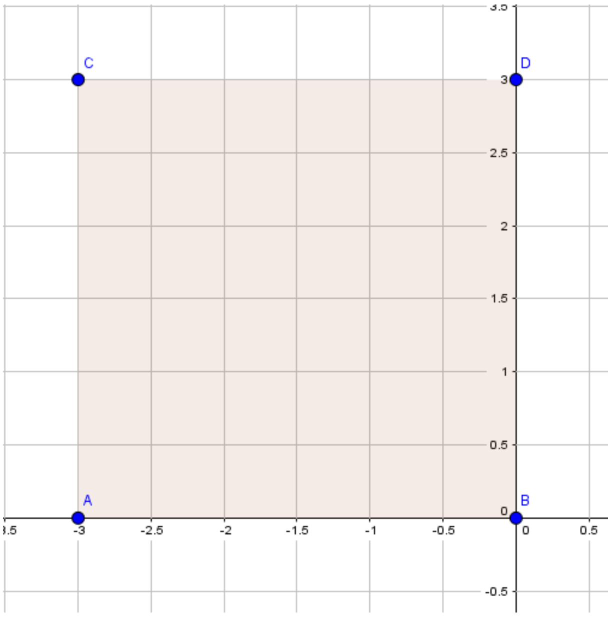
Teaching Resources:
Introductory Linear Algebra Kit and Matrix Transformation Board and their accessories (colorful board pins and string)

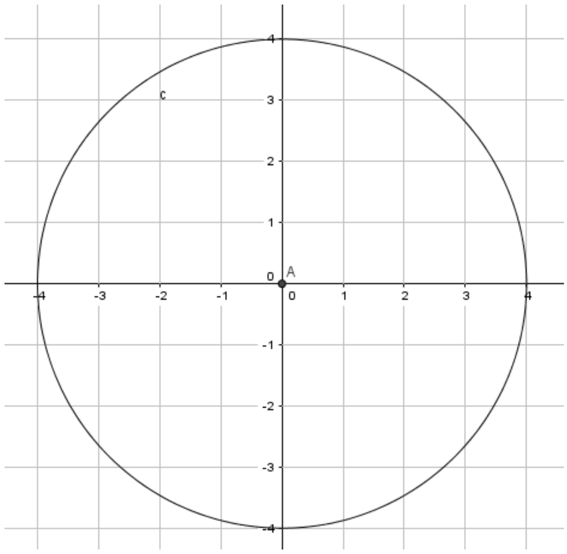
Opening of the Lesson:
The teacher shall start the lesson by posing an open-ended question for students about physical significance of matrix multiplication, determinants of matrices or even structure and set up of matrices. The teacher shall ask students if they have ever visualized or realized application of matrices in real life context. The teacher shall then discuss an example of matrix multiplication. For example $\begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} \begin{bmatrix} (1)(5)+(-3)(7) \\ (2)(5)+(4)(7) \end{bmatrix}$. The teacher would then draw students attention to the fact that matrices and linear transformations have multiple applications in business, science and engineering.

Teaching Learning Process

Instructional Strategy	**Formative Assessment**
Teacher would enquire the students about geometrical interpretation of matrices and vice versa.	Why do you think we need matrices? Do you know how to represent 2 dimensional regions like polygons, circles, ellipses, or even some irregular region with help of matrices.
The teacher would then use the Kit and Board to represent and demonstrate squares, triangles, etc. with help of pins and strings. After that, he should discuss with students how matrices are used to represent them with coordinates of their vertices going in as column vectors into the matrix. The teacher at all times shall make use of the tools to demonstrate any 2-D regions being discussed and encourage students to visualize them with help of tools as well.	

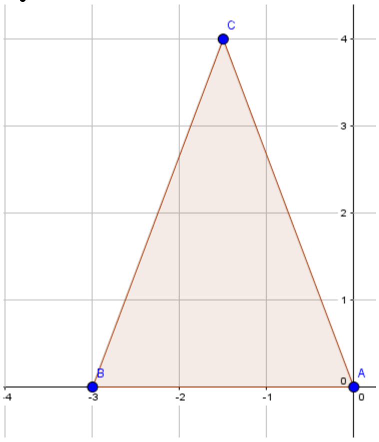
For example, a square with side 3 units placed in the II Quadrant with one vertex at the origin can be represented as $\begin{bmatrix} -3 & 0 & 0 & -3 \\ 0 & 0 & 3 & 3 \end{bmatrix}$, each column representing one vertex of the square.





Discuss several examples of regions, vectors, polygons and their representations using matrices. Move to transformations and discuss that transformations preserve some information of original regions while they transform them to having new characteristics.

The teacher shall then ask students to use the board and the kit among themselves or even try to visualize the following regions on a cartesian plane using a rough graph and find their matrix representations. A unit square in I Quadrant with one vertex at the origin. An isosceles triangle with base = 3 units and altitude = 4 units, placed in II Quadrant with one vertex at origin.



What do you will be the representation for a circle of radius 4 units with origin as centre, or some irregular blob placed anywhere on cartesian plane?

Focus on the fact that 'transformation' is just a different word for a function. It takes an input and gives some output. Specifically when we talk about transformations they take in some vector input and given out another vector. For example, $\begin{bmatrix}5\\7\end{bmatrix}$ L($\vec{v}$) $\begin{bmatrix}2\\-3\end{bmatrix}$ Transformation bring about changes in original objects while retaining some information about them as well. While talking about 2-D linear transformations, they bring about area and positional changes. They preserve the shape of original region as well. All information about 2D linear transformations are stored in four elements of the 2 $\times$ 2 standard matrix of linear transformation. We can figure a lot of information about what that transformation does just by looking at those 4 entries, while complete information can be figured out by visualizing them in action and that can be done with help of the hands-on tools.	

Lesson Plan II

Topic - Linear Transformations in Two Dimension
Sub Topic - Introduction to Linear Transformations and the Matrix of Linear Transformation

Specific Objectives:
Students would be able to conceptualize the process behind a linear transformation.
Students would be able to visualize the effect of a linear transformation on a vector by a 2 × 2 matrix.
Students would be able to identify how the standard matrix of linear transformation is computed in terms of the bases vectors.

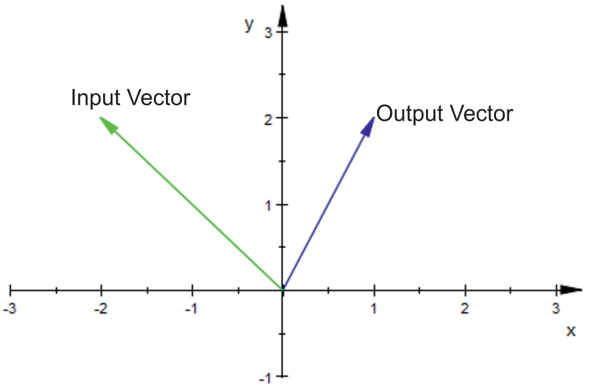
Pre Requisite Knowledge:
Students must be familiar with Coordinate Geometry & Matrix Multiplication and have knowledge about the structure of matrices. Students should be able to represent regions and vectors in 2-dimensions in terms of matrices.

Teaching Resources:
Introductory Linear Algebra Kit and Matrix Transformation Board and their accessories (board pins and string)

Opening of the Lesson:
The teacher shall open the lesson by recalling what was done in previous session. He shall now give students the tools and ask them to use them the other way round, i.e. give them some matrices and they should then represent regions with this information. Here are few matrices. $\begin{bmatrix}1 & -3 & -1\\ 2 & 4 & 0\end{bmatrix}$ $\begin{bmatrix}1 & -1 & 4 & 2\\ 1 & 4 & 1 & 4\end{bmatrix}$ $\begin{bmatrix}0 & 0 & 2 & 5 & 2\\ 0 & -2 & -3 & 1 & 2\end{bmatrix}$

Teaching Learning Process

The teacher shall discuss transformations and its aspects as given here. We use transformation at place of functions to visualize this input-output relation. The way to understand this is to imagine transformations as movements. When a transformation takes in an input vector and gives out an output vector, we imagine the input vector moving to the output vector. Tools shall be used to demonstrate the diagram below.	For the transformation, $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, what is the domain and co-domain?



And a region as a whole, we imagine all the possible input vectors in it moving to a new set of corresponding output vectors. Arbitrary transformations can look complicated and linear algebra only limits itself to special transformation called linear transformations which are easier to understand. A transformation is linear if it has two properties.

$T(0) = 0$ & $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$

Suppose the origin in the transformed region moves to the place of the point (2,-3). Will this transformation be linear?

Visually speaking this means that the origin must remain at its position and all lines must remain lines without getting curved, they must remain parallel and evenly spaced. Discuss some easy to think about linear transformations like rotation about the origin, reflection about the y axis, shear, etc.

Use the Tool to show some transformations.

Move to describing these transformations numerically.

How do you think can these transformations can be described numerically? Suppose we want to animate some graphics, to enlarge them or to squish them, what formula do we give to the computer so that if you it the coordinates of a vector, it can give you the coordinates of where that vector lands.

$\begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} \rightarrow ??? \rightarrow \begin{bmatrix} x_{out} \\ y_{out} \end{bmatrix}$

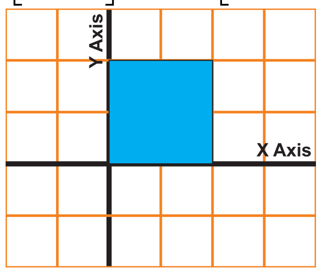
We only need to record where the two basis vector, i.e. $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ each land and everything else follows from their on. For example consider a vector $\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ meaning that it equals, $\vec{v} = -1\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$. Now if we apply some transformation on this vector, the place where $\vec{v}$ lands will be -1 times where $\hat{\mathbf{i}}$ landed and 2 times where $\hat{\mathbf{j}}$ landed. In other words it started as a certain linear combination of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ and it lands as the same linear combination of where those two vectors landed. Transformed $\vec{v}$ = -1(Transformed $\hat{\mathbf{i}}$) + 2(Transformed$\hat{\mathbf{j}}$) All this means that a 2 dimensional linear transformation is completely described by just four numbers, the two coordinates where $\hat{\mathbf{i}}$ lands and the two coordinates where $\hat{\mathbf{j}}$ lands. We package this grid of coordinates into a two-by-two grid of numbers called a 2×2 Matrix where the columns are the vectors where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ each land. Discuss the matrix $\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$ acting on $\vec{v}$ with the Kit.	Ask the students to visualize its effect using the tools.
This matrix is just a way of packaging the information needed to describe a linear transformation. Always remember to interpret the first column as where the first bases vector ($\hat{\mathbf{i}}$) lands and the second as where the second ($\hat{\mathbf{j}}$) does. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ $x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$ Draw attention of students to the fact that this is how the matrix multiplication is defined. See how it looks like a function. Discuss examples: $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ Rotate all of space 90° counter-clockwise, then where do $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ land, that gives the standard matrix for rotation by 90°. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Horizontal Shear where $\hat{\mathbf{i}}$ remains fixed so the first column is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ but $\hat{\mathbf{j}}$ moves to some other location hence the second column becomes $\begin{bmatrix} k \\ 0 \end{bmatrix}$, the value of 'k' determining the extent of how much horizontally the vector has moved. $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$	

What about going the other way around? Starting with a matrix, say with $\begin{bmatrix}1 & 3\\2 & 1\end{bmatrix}$ and we want to deduce what it's transformation looks like.

If the vectors where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ land on are linearly dependent, $\begin{bmatrix}2 & -2\\1 & -1\end{bmatrix}$, it means the linear transformation squishes all of space onto the line where those two vectors sit. (Also known as the one dimensional span of those two linearly dependent vectors.)
Demonstrate this using the Kit.

To sum up we can say, Linear Transformations are a way to move around space such that grid-lines remain parallel and evenly spaced and such that the origin remains at its place. Delightfully, these transformations can be described using only a handful (4) numbers, the coordinates of where each of the basis vector lands. Matrices give us a language to describe these transformations, where the columns represent those coordinates. And matrix multiplication is just a way to compute what that matrix does to a given vector. Every time we see a matrix, we can interpret it as a certain transformation of space.

Try to see the impact of following standard matrices on different regions. Start with a unit square with one vertex at origin, placed in I Quadrant. Can you relate these with computer graphics? $\begin{bmatrix}1 & 0\\0 & -2\end{bmatrix}$ $\begin{bmatrix}-3 & 0\\0 & 1\end{bmatrix}$ $\begin{bmatrix}-2 & 0\\0 & -2\end{bmatrix}$



Multiplying or taking composition of two matrices of linear transformation would also then result in a new linear transformation.

Lesson Plan III

Topic - Linear Transformations in Two Dimension
Sub Topic - Matrix of Linear Transformation, Significance of Determinant and Rotation Matrices

Specific Objectives:
Students would be able to conceptualize the process behind a linear transformation.
Students would be able to visualize the effect of a linear transformation on a region by a 2×2 matrix.
Students would be able to identify the role of the determinant of the standard matrix of linear transformation in changing areas of regions.
Students would be able to identify how standard matrices of rotation are computed and effects of composition of matrices

Pre Requisite Knowledge:
Students must be familiar with Coordinate Geometry & Matrix Multiplication, have conceptual idea of how 2D matrices of linear transformations change position of vectors and know the role of bases vectors while computing the standard 2×2 matrix of linear transformation.

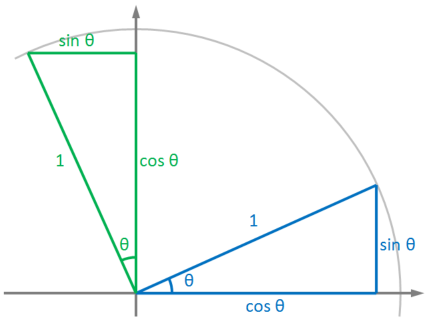
Teaching Resources:
Introductory Linear Algebra Kit and its accessories (Board Pins and String)

Opening of the Lesson:
The teacher shall start the lesson by recalling how standard matrices of 2D linear transformations are computed for vectors and how columns of standard matrix are determined with help of two examples. Examples: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ The teacher should then draw students attention to the fact that these matrices while both bring about similar changes to the unit square in I quadrant with one vertex at the origin are both different and will have different effects on other regions or vectors. While the first one is for 'reflection about the line $y = -x$', the second one is for 'reflection about origin'. This shall be discussed by making the same matrices act on the unit square in the second quadrant with one vertex at the origin.

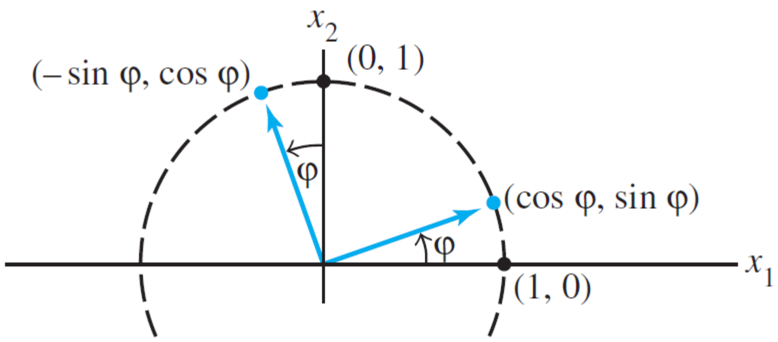
Teaching Learning Process

Instructional Strategy	**Formative Assessment**
The teacher shall write down a few matrices on the board and ask the students to find out the coordinates of the resulting quadrilateral when these matrices act as matrices of linear transformations on the unit square in the first quadrant with one vertex at the origin.	Use the following matrices as matrices of linear transformations acting on a unit square in the first quadrant with one vertex at the origin with the coordinates (0,0), (1,0), (1,1) and (0,1). Find out the coordinates of the resulting quadrilateral. $\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}$ $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ $\begin{bmatrix}0 & -3\\2 & 0\end{bmatrix}$ $\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}$ $\begin{bmatrix}1 & 2\\-1 & 2\end{bmatrix}$ Use the following matrix as matrix of linear transformations acting on a triangle with the coordinates of the vertices as (0,0), (2,0), (1,2). Find out the coordinates of the resulting triangle. $\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}$
The teacher shall write down the answers given by the students on the white board and use the kit to demonstrate the transformations and have the resulting regions stay displayed on the kit with the pins and the strings for the time being. Provide as much opportunity to students to check recheck their solutions with help of tools.	
The teacher shall then draw students' attention to the determinant of each of the standard matrices used for these transformations and simultaneously ask the students to find the areas of the resulting regions in each case with the help of the Linear Algebra Kit and Matrix Transformation Board only where the regions are displayed. The students shall themselves be able to figure out that the absolute value of the determinant in each case comes out to be equal to the corresponding areas. The teacher shall then tell the students that the determinants is the *factor* by which the area of the transformed region has increased or decreased.	Do you see any relationship between the determinants of the matrices and area of the transformed regions?
The teacher must also draw students to the fact that the matrix $\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}$ while increases the area by a factor of 6 and the matrix $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ while reflects the region by Y axis, the matrix $\begin{bmatrix}0 & -3\\2 & 0\end{bmatrix}$ which is computed by their product does both the jobs at once and is called a composition of the above two matrices.	What must be the matrix of linear transformation if we want to reflect some region about origin and then rotate it anti-clockwise by 90°?

The teacher shall now discuss with students how matrices which bring about a rotation by a fixed angle are computed. The teacher shall again take back students to how reflection matrices were determined in the previous lesson and how each column depends on where the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ vectors land. To facilitate this discussion, again the Linear Algebra Kit and Matrix Transformation Board along with writing on the board must be brought in. The image below can be demonstrated by using similar representations.



http://reedbeta.com/blog/rotations-and-infinitesimal-generators/



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After discussing the idea of rotation and effect of determinants the teacher shall then conclude the lesson by asking the students to now construct matrices with specific rotations and area changes.

Construct matrix of linear transformation which brings about a rotation of 45° anticlockwise and squishes down the area by a factor of 2.

More matrices and/or points to be discussed.

$\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$ $\begin{bmatrix} 0 & 2 \\ -1.5 & 1 \end{bmatrix}$ $\begin{bmatrix} .5 & .5 \\ -.5 & .5 \end{bmatrix}$ $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ Linearly Dependent $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Negative Determinant $\begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix}$ Negative Determinant $\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

Negative Determinant